$L^p - L^{\acute{p}}$ Estimates for the Schrödinger Equation on the Line and Inverse Scattering for the Nonlinear Schrödinger Equation with a Potential *

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Abstract

In this paper I prove a $L^p - L^{\acute{p}}$ estimate for the solutions of the one–dimensional Schrödinger equation with a potential in L^1_γ where in the *generic case* $\gamma > 3/2$ and in the *exceptional case* (i.e. when there is a half–bound state of zero energy) $\gamma > 5/2$. I use this estimate to construct the scattering operator for the nonlinear Schrödinger equation with a potential. I prove moreover, that the low–energy limit of the scattering operator uniquely determines the potential and the nonlinearity using a method that allows as well for the reconstruction of the potential and of the nonlinearity.

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1 Introduction

Let us consider the Schrödinger equation (LS)

$$i\frac{\partial}{\partial t}u(t,x)=H_0u(t,x),\ u(0,x)=\phi(x) \tag{1.1}$$

where H_0 is the self–adjoint realization of $-\Delta$ in $L^2(\mathbf{R}^n)\,, n\geq 1$,

$$H_0:=-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}. \tag{1.2}$$

The domain of $H_0, D(H_0)$, is the Sobolev space W_2. The solution to (1.1) is given by $e^{-itH_0}\phi$, where the strongly continuous unitary group e^{-itH_0} is defined by the functional calculus of self–adjoint operators. The kernel of e^{-itH_0} is given by (see Example 3 in page 59 of [24]) $(4\pi it)^{-n/2}e^{i|x-y|^2/4t}$. From this explicit expression for the kernel it follows that the restriction of e^{-itH_0} to $L^2(\mathbf{R}^n)\cap L^p(\mathbf{R}^n)$ extends to a bounded operator from $L^p(\mathbf{R}^n)$ into $L^{\acute{p}}(\mathbf{R}^n)$ such that

$$\left\|e^{-itH_0}\right\|_{\mathcal{B}\left(L^p(\mathbf{R}^n),L^{\acute{p}}(\mathbf{R}^n)\right)}\leq\frac{C}{t^{n(\frac{1}{p}-\frac{1}{2})}},\ t>0, \tag{1.3}$$

for some constant C, $1\leq p\leq 2$, and $\frac{1}{p}+\frac{1}{\acute{p}}=1$, and where for any pair of Banach spaces X,Y we denote by $\mathcal{B}(X,Y)$ the Banach space of all bounded linear operators from X into Y. In the case when $X=Y$ we use the notation $\mathcal{B}(X)$. Estimate (1.3) expresses the dispersive nature of the solutions to (1.1) and it is a fundamental tool in the study of the nonlinear Schrödinger equation:

$$i\frac{\partial}{\partial t}u=H_0u+f(u) \tag{1.4}$$

since it allows to control the nonlinear behaviour of the solutions to (1.4), that is produced by $f(u)$, in terms of the dispersion that is produced by the linear term H_0u. See for example [24], [7], [8], [9], [27], [28], [15], [16], [29], [23] and [18].

In the case of a linear Schrödinger equation with a potential (LSP):

$$i\frac{\partial}{\partial t}u(t,x)=(H_0+V)u(t,x),\ u(0,x)=\phi, \tag{1.5}$$

where V is a real–valued function defined on $\mathbf{R}^n$ such that the operator $H:=H_0+V$ is self–adjoint on $D(H_0)$, Journé, Soffer and Sogge [14] proved that for $n\geq 3$

$$\left\|e^{-itH_0}P_c\right\|_{\mathcal{B}\left(L^p(\mathbf{R}^n),L^{\acute{p}}(\mathbf{R}^n)\right)}\leq\frac{C}{t^{n(\frac{1}{p}-\frac{1}{2})}}, \tag{1.6}$$

for $1\leq p\leq 2$, $\frac{1}{p}+\frac{1}{\acute{p}}=1$ and where P_c is the orthogonal projector onto the continuous subspace of H. Note that (1.6) can not hold for the pure point subspace of H. Estimate (1.6) is the natural extension of (1.3) to the case with a potential. Besides conditions on the regularity and the decay of V (see equation (1.6) of [14]) Journé, Soffer and Sogge require that zero is neither a bound state nor a half–bound state for H. The proof given by [14] consists of a high–energy estimate that is always true and of a low–energy estimate

where the condition that zero is neither a bound state nor a half–bound state was used. The low–energy estimate of [14] was obtained by studying the behaviour near zero of the spectral family of H. For this purpose Journé, Soffer and Sogge [14] used the estimates on the behaviour near zero of the resolvent of H obtained by Jensen and Kato [13], [11] and [12] for $n \geq 3$. It is actually here that the restriction $n \geq 3$ appears in the result of [14]. One way to understand the reasons for the restriction to $n \geq 3$ is to look to the kernel of the free resolvent, $(H_0 - z)^{-1}$. For $n = 3$ this kernel is given by

$$\frac{1}{4\pi} \frac{e^{i\sqrt{z}\,|x-y|}}{|x-y|}. \tag{1.7}$$

Note that (1.7) behaves nicely as $z \to 0$. In the case $n \geq 4$ the kernel of the free resolvent has also a nice behaviour as $z \to 0$. This fact is the starting point of the analysis of Jensen and Kato in [13], [11] and [12], who use perturbation theory to estimate the behaviour near zero of the resolvent of H. In the case $n = 1$ the kernel of $(H_0 - z)^{-1}$ is given by (see Theorem 9.5.2 in page 160 of [25])

$$\frac{i}{2\sqrt{z}} e^{i\sqrt{z}|x-y|}. \tag{1.8}$$

The kernel (1.8) is singular as $z \to 0$ and an approach as in [14], [13], [11] and [12] does not appears to be convenient. We take in Section 2 below a different point of view. We base our analysis of the low–energy behaviour of the spectral family of H on the generalized Fourier maps that are constructed from the scattering solutions $\Psi_+(x,k), x, k \in \mathbf{R}$. The crucial issue here is that for $n = 1$ the construction of the scattering solutions can be reduced to the solution of Volterra integral equations. More precisely, the scattering solution is given in terms of the Jost solutions, $f_j(x,k), j = 1,2$, as follows:

$$\Psi_+(x,k) = \begin{cases} \frac{T(k)}{\sqrt{2\pi}} f_1(x,k), & k \geq 0, \\ \\ \frac{T(-k)}{\sqrt{2\pi}} f_2(x,-k), & k \leq 0, \end{cases} \tag{1.9}$$

where $T(k)$ is the transmission coefficient. The f_j are solutions to Volterra integral equations that are obtained by iteration as uniformly convergent series. See [5], [6], [3] and [2]. This fact allows for a detailed analysis of the low–energy behaviour of the spectral family of H that coupled with a high–energy estimate allows us to prove in Section 2 an estimate like (1.6) in the case $n = 1$.

Since in what follows we only consider the case $n = 1$ we denote below by $L^p, 1 \leq p \leq \infty$, the space $L^p(\mathbf{R}^1)$. For any $s \in \mathbf{R}$ let us denote by L^1_s the space of all complex–valued measurable functions, ϕ, defined on $\mathbf{R}$ such that

$$\|\phi\|_{L^1_s} := \int_{\mathbf{R}} |\phi(x)|(1+|x|)^s dx < \infty. \tag{1.10}$$

L^1_s is a Banach space with the norm (1.10). Below we always assume that $V \in L^1_1$. It follows from the existence of the Jost solutions and since the eigenvalues of $-\frac{d^2}{dx^2} + V(x)$ are simple (see [3]) that the differential expression $\tau := -\frac{d^2}{dx^2} + V(x)$ is in the limit point case at $\pm\infty$. Then by the Weyl criterion (see [32]) τ is essentially self–adjoint on the domain

$$D(\tau) := \left\{ \phi \in L^2_C : \phi \,\text{and}\, \acute{\phi} \text{ are absolutely continuous and } \tau\phi \in L^2 \right\}, \tag{1.11}$$

where we denote by $\acute{\phi}(x) = \frac{d}{dx}\phi(x)$ and by L^2_C the set of all $\phi \in L^2$ that have compact support. We denote by H the unique self–adjoint realization of τ. It is known that the absolutely continuous spectrum of H is given by $\sigma_{ac}(H) = [0, \infty)$, that H has no singular continuous spectrum, that H has no eigenvalues that are positive or equal to zero and that H has a finite number, N, of negative eigenvalues that are simple and that we denote by $-\beta_N^2 < \beta_{N-1}^2 < \cdots < -\beta_1^2 < 0$. Let F denotes the Fourier transform as a unitary operator on L^2

$$F\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\phi(x)dx. \tag{1.12}$$

We will also use the notation $\hat{\phi}(k) := F\phi(k)$. For any $\alpha \in \mathbf{R}$ let us denote by W_α the Sobolev space consisting of the completion of the Schwartz class in the norm

$$\|\phi\|_\alpha := \left\|(1+k^2)^{\alpha/2}\hat{\phi}(k)\right\|_{L^2}. \tag{1.13}$$

We denote by h the following quadratic form

$$h(\phi, \psi) := (\acute{\phi}, \acute{\psi})_{L^2} + (V\phi, \psi)_{L^2}, \tag{1.14}$$

with domain $D(h) = W_1$. Since $V \in L^1_1 \subset L^1_0 \equiv L^1$ it follows from Theorem 8.42 in page 147 of [25] and from the remarks above Theorem 9.14.1 in page 183 of [25] that h is closed and bounded from below and that the associated operator, H_h, is self–adjoint with domain, $D(H_h) \subset W_1$. Since $D(\tau) \subset W_1$ it follows that H_h is a self–adjoint extension of τ and as τ is essentially self–adjoint we have that $H = H_h$ and then $D(|H|) = W_1$. For u, v any pair of solutions to the stationary Schrödinger equation:

$$-\frac{d^2}{dx^2}u + Vu = k^2u, \; k \in \mathbf{R}, \tag{1.15}$$

let $[u, v]$ denotes the Wronskian of u and v:

$$[u, v] := \acute{u}v - u\acute{v}. \tag{1.16}$$

A potential V is said to be *generic* if the Jost solutions at zero energy satisfy $[f_1(x, 0), f_2(x, 0)] \neq 0$ and V is said to be *exceptional* if $[f_1(x, 0), f_2(x, 0)] = 0$. If the potential V is exceptional there is a bounded solution (a half–bound state) to the equation (1.15) with $k = 0$. See [21] for these definitions and a discussion of related issues. Let P_c denotes the projector onto the continuous subspace of H. Note that $P_c = I - P_p$, where P_p is the projector onto the finite dimensional subspace of L^2 generated by the eigenvectors corresponding to the N eigenvalues of H.

Our mail result is the following theorem that we prove in Section 2.

THEOREM 1.1. *(The $L^1 - L^\infty$ estimate). Suppose that $V \in L^1_\gamma$ where in the* generic case $\gamma > 3/2$ *and in the* exceptional case $\gamma > 5/2$. *Then for some constant C*

$$\left\|e^{-itH}P_c\right\|_{\mathcal{B}(L^1, L^\infty)} \leq \frac{C}{\sqrt{t}}, \; t > 0. \tag{1.17}$$

COROLLARY 1.2. *(The $L^p - L^{\acute{p}}$ estimate). Suppose that the conditions of Theorem 1.1 are satisfied. Then for* $1 \leq p \leq 2$ *and* $\frac{1}{p} + \frac{1}{\acute{p}} = 1$

$$\left\| e^{-itH} P_c \right\|_{\mathcal{B}\left(L^p, L^{\acute{p}}\right)} \leq \frac{C}{t^{\left(\frac{1}{p}-\frac{1}{2}\right)}},\ t > 0. \tag{1.18}$$

COROLLARY 1.3. *(The espace–time estimate). Suppose that the conditions of Theorem 1.1 are satisfied. Then*
(a)

$$e^{-itH} P_c \in \mathcal{B}\left(L^2, L^6(\mathbf{R} \times \mathbf{R})\right). \tag{1.19}$$

(b) If moreover, H has no negative eigenvalues and

$$i\frac{\partial}{\partial t} u(t,x) = H u(t,x) + g(t,x),\ u(0,x) = f(x), \tag{1.20}$$

then

$$\|u(t,x)\|_{L^6(\mathbf{R}\times\mathbf{R})} \leq C\left[\|f\|_{L^2} + \|g\|_{L^{6/5}(\mathbf{R}\times\mathbf{R})}\right]. \tag{1.21}$$

In the case $V = 0$ and $n \geq 1$ Theorem 1.1 and Corollaries 1.2 and 1.3 were proven by Strichartz in [30]. They were proven in [14] for $n \geq 3$ and V satisfying appropriate conditions on regularity and decay (see [14], equation (1.6)). In [14] it was assumed moreover, that zero is neither a bound state nor a half–bound state. Note that we do not have to assume that zero is not a half–bound state for Theorem 1.1 and Corollaries 1.2 and 1.3 to hold. In our case it is enough to require that V has a slightly faster decay at infinity when there is a half–bound state at zero.

Theorem 1.1 and Corollaries 1.2 and 1.3 open the way to the study of the scattering theory for the nonlinear Schrödinger equation with a potential (NLSP):

$$i\frac{\partial}{\partial t} u = H u + f(|u|)\frac{u}{|u|}. \tag{1.22}$$

As a first application we study in this paper the low–energy scattering for the NLSP and we prove that the low–energy limit of the scattering operator uniquely determines the potential and the nonlinearity. For this purpose we proceed as in [31] were the case $n \geq 3$ was considered. Let us assume that H has no negative eigenvalues. Then $H > 0$ and since $D(\sqrt{H}) = W_1$ the operators $\sqrt{H+1}\,(-\Delta+1)^{-1/2}$ and $\sqrt{-\Delta+1}\,(H+1)^{-1/2}$ are bounded in L^2. It follows that the norm associated to the following scalar product

$$(\phi, \psi)_X := \left(\sqrt{H+1}\,\phi, \sqrt{H+1}\,\psi\right)_{L^2}, \tag{1.23}$$

is equivalent to the norm of W_1. We denote by X the Sobolev space W_1 endowed with the scalar product (1.23). The space X is a Hilbert space. Clearly, e^{-itH} is a strongly continuous group of unitary operators on X. For any $\delta > 0$ we denote:

$$X(\delta) := \{\phi \in X : \|\phi\|_X < \delta\}. \tag{1.24}$$

Let us denote $X_3 := L^{p+1}$ and $r = (p-1)/(1-d)$ with $d := \frac{1}{2}(p-1)/(p+1)$ and $5 \leq p < \infty$. In what follows for functions $u(t,x)$ defined on $\mathbf{R} \times \mathbf{R}$ we write $u(t)$ for $u(t,\cdot)$.

THEOREM 1.4. *(Low–energy scattering). Suppose that $V \in L^1_\gamma$ where in the* generic case *$\gamma > 3/2$ and in the* exceptional case *$\gamma > 5/2$ and that H has no negative eigenvalues. Assume moreover, that the function f in (1.22) is defined on $\mathbf{R}$, that it is real–valued and C^1. Furthermore, $f(0) = 0$ and*

$$\left| \frac{d}{d\mu} f(\mu) \right| \leq C|\mu|^{p-1}, \tag{1.25}$$

for some $5 \leq p < \infty$. Then there is a $\delta > 0$ such that for every $\phi_- \in X(\delta)$ there is a unique solution to the NLSP, $u(t,x)$, such that $u \in C(\mathbf{R}, X) \cap L^r(\mathbf{R}, X_3)$ and

$$\lim_{t \to -\infty} \left\| u(t) - e^{-itH} \phi_- \right\|_X = 0. \tag{1.26}$$

Moreover, there exists a unique $\phi_+ \in X$ such that

$$\lim_{t \to \infty} \left\| u(t) - e^{-itH} \phi_+ \right\|_X = 0. \tag{1.27}$$

For all $t \in \mathbf{R}$

$$\frac{1}{2}\|u(t)\|_X^2 + \int_{\mathbf{R}} F(|u(t)|)dx = \frac{1}{2}\|\phi_-\|_X^2 = \frac{1}{2}\|\phi_+\|_X^2, \tag{1.28}$$

where F is the primitive of f such that $F(0) = 0$. In addition the nonlinear scattering operator $S_V : \phi_- \to \phi_+$ is a homeomorphism from $X(\delta)$ onto $X(\delta)$.

Theorem 1.4 is proven in Section 3 using Theorem 1.1, Corollaries 1.2 and 1.3 and the abstract low–energy scattering theory of Strauss [27], [28]. The scattering operator S_V compares solutions of the NLSP (1.22) with solutions to the LSP (1.5). To reconstruct V we consider below the scattering operator, S, that compares solutions to the NLSP with solutions to the LS (1.1). For this purpose let us consider the wave operators

$$W_\pm := \mathrm{s} - \lim_{t \to \pm\infty} e^{itH} \, e^{-itH_0}. \tag{1.29}$$

The $W_\pm$ are unitary on L^2 (note that H has no eigenvalues). The existence of the strong limits in (1.29) is well known (see Theorem 9.14.1 in page 183 of[25]). Moreover, by the intertwining relations, $\sqrt{H} W_\pm = W_\pm \sqrt{H_0}$ and as $D(\sqrt{H}) = W_1$, we have that $W_\pm$ and $W^*_\pm$ belong to $\mathcal{B}(W_1)$ and for $0 < \delta_1 < \delta$ they send $X(\delta_1)$ into $X(\delta)$ if δ_1 is small enough. Let us define:

$$S := W_+^* S_V W_-. \tag{1.30}$$

Take δ_1 so small that $W_- X(\delta_1) \subset X(\delta)$ with δ as in Theorem 1.4 and then δ_2 so large that $W_+^* X(\delta) \subset X(\delta_2)$. Then S sends $X(\delta_1)$ into $X(\delta_2)$. Moreover, for any $\psi_- \in X(\delta_1)$ let us take in Theorem 1.4 $\phi_- \equiv W_- \psi_-$ and let $u(t,x)$ and ϕ_+ be as in Theorem 1.4. Let us denote $\psi_+ := S\psi_- = W_+^* \phi_+$. Then by Theorem 1.4 and (1.29)

$$\lim_{t \to \pm\infty} \left\| u(t,x) - e^{-itH_0} \psi_\pm \right\|_{L^2} = 0. \tag{1.31}$$

That is to say, S sends the initial data at $t = 0, \psi_-$, of the incoming solution to LS to the initial data at $t = 0, \psi_+$, of the outgoing solution to LS. Let us denote by S_L the linear scattering operator corresponding to the LS and the LSP:

$$S_L := W_+^* \, W_-. \tag{1.32}$$

In Theorem 1.5 below, S_L is reconstructed from the low–energy limit of S.

THEOREM 1.5. *Suppose that the assumptions of Theorem 1.4 are satisfied. Then for every* $\phi, \psi \in X$

$$\lim_{\epsilon \downarrow 0} \left(S\epsilon\phi, \psi\right)_{L^2} = \left(S_L\phi, \psi\right)_{L^2}. \tag{1.33}$$

Since, as is well known, from S_L we can uniquely reconstruct V we obtain the following Corollary.

COROLLARY 1.6. *Suppose that the assumptions of Theorem 1.5 are satisfied. Then the scattering operator, S, uniquely determines the potential* V*.*

In the case where $f(u) = \lambda |u|^p$, we can also uniquely reconstruct the coupling constant λ.

COROLLARY 1.7. *Suppose that the assumptions of Theorem 1.4 are satisfied and that moreover,* $f(u) = \lambda |u|^p$*, for some constant* λ*. Then the scattering operator, S, uniquely determines the potential* V *and the coupling constant* λ*. Furthermore, for all* $0 \neq \phi \in X \cap L^{1+\frac{1}{p}}$*:*

$$\lambda = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon^p} \frac{\left((S_V - I)\epsilon\phi, \phi\right)_{L^2}}{\int_{-\infty}^{\infty} \|e^{-itH}\phi\|_{L^{1+p}}^{1+p}}. \tag{1.34}$$

Remark that by Sobolev's imbedding theorem [1], $X \subset L^{1+p}$. Then by (1.18)

$$0 < \int_{-\infty}^{\infty} \left\| e^{-itH}\phi \right\|_{L^{1+p}}^{1+p} dt < \infty. \tag{1.35}$$

Theorem 1.5 and Corollaries 1.6 and 1.7 are proven as in [31] (see Section 3).

We use below the letter C to denote any positive constant whose particular value is not relevant.

2 The $L^p - L^{\acute{p}}$ Estimate

We assume that $V \in L^1_1$. For any complex number, k, we denote by $\Re k$ and $\Im k$, respectively, the real and the imaginary parts of k. The Jost solutions $f_j(x,k), j = 1,2$, are solutions to the stationary Schrödinger equation

$$-\frac{d^2}{dx^2} f_j(x,k) + V(x) f_j(x,k) = k^2 f_j(x,k) \tag{2.1}$$

were $\Im k \geq 0$. To construct the Jost solution we define $m_1(x,k) := e^{-ikx} f_1(x,k)$ and $m_2(x,k) := e^{ikx} f_2(x,k)$. They are, respectively, solutions of the following equations:

$$\frac{d^2}{dx^2} m_1(x,k) + 2ik \frac{d}{dx} m_1(x,k) = V(x) m_1(x,k), \tag{2.2}$$

$$\frac{d^2}{dx^2} m_2(x,k) - 2ik \frac{d}{dx} m_2(x,k) = V(x) m_2(x,k). \tag{2.3}$$

The $m_j(x,k), j = 1,2$, are the unique solutions of the Volterra integral equations

$$m_1(x,k) = 1 + \int_x^{\infty} D_k(y-x) V(y) m_1(y,k) dy, \tag{2.4}$$

$$m_2(x,k) = 1 + \int_{-\infty}^{x} D_k(x-y)V(y)m_2(y,k)dy, \tag{2.5}$$

where

$$D_k(x) := \int_0^x e^{2iky}dy = \begin{cases} \frac{1}{2ik}(e^{2ikx}-1), & k \neq 0, \\ x, & k = 0. \end{cases} \tag{2.6}$$

Note that $f_1(x,k) \sim e^{ikx}$ as $x \to \infty$ and that $f_2(x,k) \sim e^{-ikx}$ as $x \to -\infty$. A detailed study of the properties of the $m_j(x,k), j = 1,2,$ was carried over in [3]. Here we state a number of results from [3] that we need. In what follows we denote by C any positive constant whose specific value is not relevant to us and by $\dot{g}(x,k) := \frac{\partial}{\partial k}g(x,k)$. For each fixed $x \in \mathbf{R}$ the $m_j(x,k)$ are analytic in k for $\Im k > 0$ and continuous in $\Im k \geq 0$ and

$$|m_1(x,k) - 1| \leq C\, \frac{1+\max(-x,0)}{1+|k|}, \tag{2.7}$$

$$|m_2(x,k) - 1| \leq C\, \frac{1+\max(x,0)}{1+|k|}. \tag{2.8}$$

Moreover, $\dot{m}_j(x,k), j = 1,2,$ exits for $\Im k \geq 0, k \neq 0$, $k\dot{m}_j(x,k)$ is continuous in k for each fixed $x \in \mathbf{R}$ and for each fixed $x_0 \in \mathbf{R}$ there is a constant C_{x_0} such that

$$|\dot{m}_1(x,k)| \leq C_{x_0}\, \frac{1}{|k|}, x \geq x_0, \tag{2.9}$$

$$|\dot{m}_2(x,k)| \leq C_{x_0}\, \frac{1}{|k|}, x \leq x_0. \tag{2.10}$$

In the Lemma below we slighly improve the estimates (2.9) and (2.10) under the assumption that $V \in L^1_\gamma$ for $1 < \gamma \leq 2$.

LEMMA 2.1. *Suppose that $V \in L^1_\gamma$ for some $1 \leq \gamma \leq 2$. Then for each $x_0 \in \mathbf{R}$ there is a constant C_{x_0} such that*

$$|\dot{m}_1(x,k)| \leq C_{x_0}\, \frac{|k|^\gamma}{|k|^2(1+|k|)^{\gamma-1}}, x \geq x_0, \tag{2.11}$$

$$|\dot{m}_2(x,k)| \leq C_{x_0}\, \frac{|k|^\gamma}{|k|^2(1+|k|)^{\gamma-1}}, x \leq x_0. \tag{2.12}$$

Proof : We give the proof in the case of $\dot{m}_1(x,k)$. The case of $\dot{m}_2(x,k)$ follows similarly. It follows from (2.6) that for $k \neq 0$

$$\left|\dot{D}_k(x)\right| = \left|\frac{1}{k}\int_0^x y\left(\frac{\partial}{\partial y}e^{2iky}\right)dy\right| \leq 2\frac{|x|}{|k|}, \tag{2.13}$$

and that

$$\left|\dot{D}_k(x)\right| \leq |x|^2. \tag{2.14}$$

By (2.13) and (2.14) for any $1 \leq \gamma \leq 2$

$$|\dot{D}_k(x)| \leq \frac{2^{2-\gamma}|x|^\gamma}{|k|^{2-\gamma}}. \tag{2.15}$$

Since (2.4) is a Volterra integral equation, $m_1(x,k)$ is obtained by iteration [3]:

$$m_1(x,k) = \lim_{n\to\infty} m_{1,n}(x,k), \tag{2.16}$$

where $m_{1,0}(x,k) = 1$ and for $n = 1, 2, \cdots$

$$m_{1,n}(x,k) = 1 + \sum_{l=1}^{n} g_l(x,k), \tag{2.17}$$

where

$$g_l(x,k) = \int_{x\leq x_1\leq x_2\leq\cdots\leq x_l} D_k(x_1 - x)D_k(x_2 - x_1)\cdots D_k(x_l - x_{l-1})V(x_1)\cdots V(x_l)dx_1\cdots dx_l. \tag{2.18}$$

Moreover, the $m_{1,n}$ satisfy the following equation for $n = 0, 1, \cdots$

$$m_{1,n+1}(x,k) = 1 + \int_x^\infty D_k(y - x)V(y)m_n(y,k)dy. \tag{2.19}$$

Then,

$$\dot{m}_{1,n+1}(x,k) = \int_x^\infty \dot{D}_k(y - x)V(y)m_n(y,k)dy + \int_x^\infty D_k(y - x)V(y)\dot{m}_n(y,k)dy. \tag{2.20}$$

Furthermore, since by (2.6)

$$|D_k(x)| \leq |x|, \tag{2.21}$$

it follows from (2.18) that

$$|g_l(x,k)| \leq \frac{1}{l!}\left(\int_x^\infty (y - x)V(y)dy\right)^l, \tag{2.22}$$

and then by (2.17) for $x \geq x_0$

$$\begin{aligned} |m_{1,n}(x,k)| &\leq 1 + \sum_{l=1}^{n} \frac{1}{l!}\left(\int_x^\infty (y - x)|V(y)|dy\right)^l \\ &\leq e^{\left(\int_x^\infty (|x_0|+|y|)|V(y)|dy\right)},\ x \geq x_0. \end{aligned} \tag{2.23}$$

We can now estimate the first integral in the right–hand side of (2.20) as follows

$$\begin{aligned} &\left|\int_x^\infty \dot{D}_k(y - x)V(y)m_n(y,k)dy\right| \leq \\ &\frac{2^{2-\gamma}}{|k|^{2-\gamma}}\int_x^\infty |y - x|^\gamma\, |V(y)|e^{\int_x^\infty (|x_0|+|y|)|V(y)|dy} \leq C\,\frac{1}{|k|^{2-\gamma}}, x \geq x_0, \end{aligned} \tag{2.24}$$

where we used (2.15). Then using again (2.20) and (2.21) we obtain that

$$|\dot{m}_{1,n+1}(x,k)| \leq \frac{C}{|k|^{2-\gamma}} + \int_x^\infty |y - x||V(y)||\dot{m}_n(y,k)|dy. \tag{2.25}$$

Since $m_0(y,k) \equiv 1$ it follows from (2.25) with $n = 0$ that

$$|\dot{m}_{1,1}(x,k)| \leq \frac{C}{|k|^{2-\gamma}}. \tag{2.26}$$

Then by (2.25) we have that

$$|\dot{m}_{1,2}(x,k)| \leq \frac{C}{|k|^{2-\gamma}}(1+q(x)), x \geq x_0, \tag{2.27}$$

where

$$q(x) := \int_x^{\infty} (|x|_0 + |y|)|V(y)|dy, \tag{2.28}$$

and then, iterating (2.25) $n-1$ more times we prove that

$$|\dot{m}_{1,n+1}(x,k)| \leq \frac{C}{|k|^{2-\gamma}} \sum_{l=0}^{n} \frac{(q(x))^l}{l!}. \tag{2.29}$$

Taking the limit as $n \to \infty$ in (2.29) we prove that

$$|\dot{m}_1(x,k)| \leq \frac{C}{|k|^{2-\gamma}} e^{q(x)}, x \geq x_0. \tag{2.30}$$

Since $V \in L^1_{\gamma} \subset L^1_1$, we can take $\gamma = 1$ in (2.30) and then

$$|\dot{m}_1(x,k)| \leq \frac{C}{|k|}\, e^{q(x)},\ x \geq x_0. \tag{2.31}$$

Equation (2.11) follows from (2.30) and (2.31).

COROLLARY 2.2. *Suppose that $V \in L^1_{\gamma}$, for some $1 \leq \gamma \leq 2$. Then for each $x_0 \in \mathbf{R}$ there is a constant C_{x_0} such that for all $\Im k \geq 0$*

$$\left|\dot{m}'_1(x,k)\right| \leq C_{x_0}\left[1 + \frac{|k|^{\gamma}}{|k|^2(1+|k|)^{\gamma-1}}\right], x \geq x_0, \tag{2.32}$$

$$\left|\dot{m}'_2(x,k)\right| \leq C_{x_0}\left[1 + \frac{|k|^{\gamma}}{|k|^2(1+|k|)^{\gamma-1}}\right],\ x \leq x_0. \tag{2.33}$$

Proof : We prove (2.32). The proof of (2.33) is similar. By (2.4) and (2.6)

$$m'_1(x,k) = -\int_x^{\infty} e^{2ik(y-x)} V(y) m_1(y,k) dy, \tag{2.34}$$

and then

$$\dot{m}'_1(x,k) = -\int_x^{\infty} \left[2ie^{2ik(y-x)}(y-x)V(y)m_1(y,k) + e^{2ik(y-x)}V(y)\dot{m}_1(y,k)\right] dy. \tag{2.35}$$

It follows from (2.7), (2.11) and (2.35) that

$$\left|\dot{m}'_1(x,k)\right| \leq C_{x_0}\left[1 + \frac{|k|^{\gamma}}{|k|^2(1+|k|^{\gamma-1})}\right], x \geq x_0. \tag{2.36}$$

LEMMA 2.3. *Suppose that $V \in L^1_\gamma$, for some $2 \leq \gamma \leq 3$. Then for every $x_0 \in \mathbf{R}$ there is a constant C_{x_0} such that*

$$|\dot{m}_1(x,k) - \dot{m}_1(x,0)| \leq C_{x_0}|k|^{\gamma-2}, x \geq x_0, \tag{2.37}$$

$$|\dot{m}_2(x,k) - \dot{m}_2(x,0)| \leq C_{x_0}|k|^{\gamma-2}, x \leq x_0. \tag{2.38}$$

Proof: It follows from the definition of $D_k(x)$ in (2.6) that

$$\left|\dot{D}_k(x) - \dot{D}_0(x)\right| \leq \frac{4}{3}|k||x|^3, \tag{2.39}$$

and that

$$\left|\dot{D}_k(x) - \dot{D}_0(x)\right| \leq 2|x|^2. \tag{2.40}$$

Then for any $2 \leq \gamma \leq 3$ there is a constant, C_γ, such that

$$\left|\dot{D}_k(x) - \dot{D}_0(x)\right| \leq C_\gamma |k|^{\gamma-2}|x|^\gamma. \tag{2.41}$$

We obtain from (2.20) that

$$\dot{m}_{1,n+1}(x,k) - \dot{m}_{1,n+1}(x,0) = \int_x^\infty \left[\dot{D}_k(y-x) - \dot{D}_0(y-x)\right] V(y) m_n(y,k) dy +$$

$$\int_x^\infty \left\{\dot{D}_0(y-x)V(y)\left[m_n(y,k) - m_n(y,0)\right] + \left[D_k(y-x) - D_0(y-x)\right] V(y)\dot{m}_n(y,k)\right\} dy +$$

$$\int_x^\infty D_0(y-x)V(y)\left[\dot{m}_n(y,k) - \dot{m}_n(y,0)\right] dy. \tag{2.42}$$

Moreover, by (2.23) and (2.41)

$$\left|\int_0^\infty \left[\dot{D}_k(y-x) - \dot{D}_0(y-x)\right] V(y) m_n(y,k) dy\right| \leq C_{x_0}|k|^{\gamma-2}, x \geq x_0. \tag{2.43}$$

By (2.29) with $\gamma = 2$

$$|m_{1,n}(x,k) - m_{1,n}(x,0)| = \left|\int_0^k \dot{m}_{1,n}(x,s) ds\right| \leq C_{x_0}\,|k|, x \geq x_0, \tag{2.44}$$

and then by (2.14)

$$\left|\int_x^\infty \dot{D}_0(y-x)V(y)\left[m_n(y,k) - m_n(y,0)\right] dy\right| \leq C_{x_0}\,|k|,\, x \geq x_0. \tag{2.45}$$

Moreover, by (2.6)

$$|D_k(y) - D_0(y)| \leq |k||y|^2, \tag{2.46}$$

and it follows from (2.29) with $\gamma = 2$ that

$$\left|\int_x^\infty \left[D_k(y-x) - D_0(y-x)\right] V(y)\dot{m}_n(y,k) dk\right| \leq C_{x_0}|k|, x \geq x_0. \tag{2.47}$$

Then we obtain from (2.21), (2.42), (2.43), (2.45) and (2.47) that for $|k| \leq 1$:

$$|\dot{m}_{n+1}(x,k) - \dot{m}_{n+1}(x,0)| \leq C_{x_0}|k|^{\gamma-2} +$$

$$\int_x^\infty (y-x)|V(y)|\,|\dot m_n(y,k)-\dot m_n(y,0)|\,dy,\, x\geq x_0. \tag{2.48}$$

But since $m_0(x,k)\equiv 1$ it follows from (2.48) with $n=0$ that

$$|\dot m_{1,1}(x,k)-\dot m_{1,1}(x,0)|\leq C_{x_0}|k|^{\gamma-2}. \tag{2.49}$$

Iterating (2.48) n more times we prove that

$$|\dot m_{1,n+1}(x,k)-\dot m_{1,n+1}(x,0)|\leq C_{x_0}|k|^{\gamma-2}\left(1+\sum_{l=1}^{n}\frac{(q(x))^l}{l!}\right), \tag{2.50}$$

with $q(x)$ as in (2.28) and taking the limit as $n\to\infty$ we have that

$$|\dot m_1(x,k)-\dot m_1(x,0)|\leq C_{x_0}|k|^{\gamma-2}e^{q(x)},\, x\geq x_0, \tag{2.51}$$

and this proves (2.37). Equation (2.38) follows similarly.

COROLLARY 2.4. *Suppose that $V\in L^1_\gamma$ for some $2\leq\gamma\leq 3$. Then for every $x_0\in\mathbf{R}$ there is a constant C_{x_0} such that*

$$\left|\dot m_1(x,k)-\dot m_1(x,0)\right|\leq C_{x_0}\,|k|^{\gamma-2},\, x\geq x_0, \tag{2.52}$$

$$\left|\dot m_2(x,k)-\dot m_2(x,0)\right|\leq C_{x_0}\,|k|^{\gamma-2},\, x\leq x_0. \tag{2.53}$$

Proof: We give the proof of (2.52). Equation (2.53) follows in a similar way. By (2.35)

$$\dot m_1(x,k)-\dot m_1(x,0)=-\int_x^\infty dy\,\left[e^{2ik(y-x)}-1\right]V(y)\left\{2i(y-x)m_1(y,k)+\dot m_1(y,k)\right\}-$$

$$\int_x^\infty dy\,V(y)\left[2i(y-x)\left(m_1(y,k)-m_1(y,0)\right)+\dot m_1(y,k)-\dot m_1(y,0)\right]dy. \tag{2.54}$$

Then by (2.7), (2.11)with $\gamma=2$ and (2.37)

$$\left|\dot m_1(x,k)-\dot m_1(x,0)\right|\leq C_{x_0}\,|k|^{\gamma-2},\, x\geq x_0. \tag{2.55}$$

■

The Jost solutions, $f_j(x,k), j=1,2$, are independent solutions to (2.1) for $k\neq 0$ and there are unique functions $T(k)$ and $R_j(k), j=1,2$, such that [3]

$$f_2(x,k)=\frac{R_1(k)}{T(k)}f_1(x,k)+\frac{1}{T(k)}f_1(x,-k), \tag{2.56}$$

$$f_1(x,k)=\frac{R_2(k)}{T(k)}f_2(x,k)+\frac{1}{T(k)}f_2(x,-k), \tag{2.57}$$

for $k\in\mathbf{R}\setminus 0$. The function $T(k)f_1(x,k)$ describes the scattering from left to right of a plane wave e^{ikx} and $T(k)f_2(x,k)$ describes the scattering from right to left of a plane wave e^{-ikx}. The function $T(k)$ is the transmission coefficient , $R_2(k)$ is the reflection coefficient

from left to right and $R_1(k)$ is the reflection coefficient from right to left. The relations (2.56) and (2.57) are expressed as follows in terms of the $m_j(x,k), j=1,2$,

$$T(k)m_2(x,k) = R_1(k)e^{2ikx}m_1(x,k) + m_1(x,-k), \tag{2.58}$$

$$T(k)m_1(x,k) = R_2(k)e^{-2ik}m_2(x,k) + m_2(x,-k). \tag{2.59}$$

Moreover, $T(k)$ is meromorphic for $\Im k > 0$ with a finite number of simple poles, $i\beta_N, i\beta_{N-1}, \cdots, i\beta_1$, $\beta_j > 0$, $j = 1,2,\cdots,N$, on the imaginary axis. The numbers, $-\beta_N^2$, $-\beta_{N-1}^2, \cdots, -\beta_1^2$, are the simple eigenvalues of H. Furthermore, $T(k)$ is continuous in $\Im k \geq 0, k \neq i\beta_1, i\beta_2, \cdots i\beta_N$ and $T(k) \neq 0$ for $k \neq 0$. the $R_j(k), j=1,2$, are continuous for $k \in \mathbf{R}$. Moreover, the following formulas hold [3]

$$\frac{1}{T(k)} = \frac{1}{2ik}[f_1(x,k), f_2(x,k)] = 1 - \frac{1}{2ik}\int_{-\infty}^{\infty} V(y)\, m_j(y,k)\, dk,\ j=1,2. \tag{2.60}$$

$$\frac{R_1(k)}{T(k)} = \frac{1}{2ik}[f_2(x,k), f_1(x,-k)] = \frac{1}{2ik}\int_{-\infty}^{\infty} e^{-2iky}\, V(y)\, m_2(y,k)\, dy, \tag{2.61}$$

$$\frac{R_2(k)}{T(k)} = \frac{1}{2ik}[f_2(x,-k), f_1(x,k)] = \frac{1}{2ik}\int_{-\infty}^{\infty} e^{2iky} V(y)\, m_1(y,k)\, dy. \tag{2.62}$$

Furthermore,

$$T(k) = 1 + O\left(\frac{1}{|k|}\right),\ |k| \to \infty, \Im k \geq 0, \tag{2.63}$$

$$R_j(k) = O\left(\frac{1}{|k|}\right), |k| \to \infty, k \in \mathbf{R}, \tag{2.64}$$

and

$$|T(k)|^2 + |R_j(k)|^2 = 1,\ j=1,2,\ k \in \mathbf{R}. \tag{2.65}$$

The behaviour as $k \to 0$ is as follows:
(a) In the *generic case*

$$T(k) = \alpha k + o(k),\ \alpha \neq 0, k \to 0, \Im k \geq 0, \tag{2.66}$$

and $R_1(0) = R_2(0) = -1$.
(b) In the *exceptional case*

$$T(k) = \frac{2a}{1+a^2} + o(1),\ k \to 0, \Im k \geq 0, \tag{2.67}$$

$$R_1(k) = \frac{1-a^2}{1+a^2} + o(1),\ k \to 0,\ k \in \mathbf{R}, \tag{2.68}$$

$$R_2(k) = \frac{a^2-1}{1+a^2} + o(1),\ k \to 0,\ k \in \mathbf{R}, \tag{2.69}$$

where $a = \lim_{x\to-\infty} f_1(x,0) \neq 0$. For the results above about $T(k)$ and $R_j(k)$, $j=1,2$, see [3], [21] and [17]. In particular for the continuity of $T(k)$ and of $R_j(k)$ as $k \to 0$ in the exceptional case for $V \in L_1^1$ see [17].

THEOREM 2.5. *Assume that* $V \in L^1_\gamma$.
(a)If V *is* generic *and* $1 \le \gamma \le 2$, *then*

$$|\dot{T}(k)| \le C(1+|k|)^{-1},\ \Im k \ge 0, \tag{2.70}$$

$$R_j(k_1) - R_j(k_2) = \begin{cases} o\left(|k_1-k_2|^{\gamma-1}\right), & 1 \le \gamma < 2, \\ O(|k_1-k_2|), & \gamma = 2, \end{cases} \tag{2.71}$$

as $k_1 - k_2 \to 0$.
(b) If V *is* exceptional *and* $2 \le \gamma \le 3$, *then*

$$|\dot{T}(k)| \le C\frac{|k|^{\gamma-3}}{(1+|k|)^{\gamma-2}}, \tag{2.72}$$

$$T(k) - T(0) = O(|k|), k \to 0, \tag{2.73}$$

$$R_j(k) - R_j(0) = O(|k|),\ k \to 0,\ j = 1,2. \tag{2.74}$$

Moreover, if $\gamma > 2$

$$R_j(k_1) - R_j(k_2) = O\left(|k_1-k_2|^{\gamma-2}\right),\ k_1 - k_2 \to 0. \tag{2.75}$$

Proof: It follows from (2.7) and (2.34) that

$$|\acute{m}_1(x,k)| \le C,\ x \in \mathbf{R},\ \Im k \ge 0. \tag{2.76}$$

We similarly prove that

$$|\acute{m}_2(x,k)| \le C,\ x \in \mathbf{R},\ \Im k \ge 0. \tag{2.77}$$

Then (2.70) follows from (2.7), (2.8), (2.11), (2.12), (2.32), (2.33), the first equality in (2.60), (2.63), (2.66), (2.76) and (2.77).

If follows from (2.19) that

$$m_{1,n+1}(x,k_1) - m_{1,n+1}(x,k_2) = f_n(x,k_1,k_2) + \int_x^\infty D_{k_2}(y-x)V(y)$$
$$[m_{1,n}(y,k_1) - m_{1,n}(y,k_2)]\,dy, \tag{2.78}$$

where

$$f_n(x,k_1,k_2) := \int_x^\infty \left[D_{k_1}(y-x) - D_{k_2}(y-x)\right] V(y) m_{1,n}(y,k_1)\,dy. \tag{2.79}$$

Moreover, by (2.6)

$$|D_{k_1}(x) - D_{k_2}(x)| \le 2\frac{|k_1-k_2||x|}{1+|k_1-k_2||x|}|x|. \tag{2.80}$$

Then by (2.23) for $x \ge 0$

$$|f_n(x,k_1,k_2)| \le f_\gamma(k_1-k_2), \tag{2.81}$$

where for $1 \le \gamma \le 2$

$$f_\gamma(k) = C\,|k|^{\gamma-1}\int_0^\infty y^\gamma |V(y)| \left(\frac{|k|y}{1+|k|y}\right)^{2-\gamma} dy. \tag{2.82}$$

Note that as $k \to 0$

$$f_\gamma(k) = \begin{cases} o\left(|k|^{\gamma-1}\right), & 1 \le \gamma < 2, \\ \\ O\left(|k|\right), & \gamma = 2. \end{cases} \tag{2.83}$$

Since the function: $\lambda \to |k|\lambda(1+|k|\lambda)^{-1}$ is an increasing function of λ, for $\lambda \ge 0$, we have that (see (2.7) and (2.80)) for all $x \in \mathbf{R}$

$$\int_0^\infty |D_{k_1}(y-x) - D_{k_2}(y-x)|\,|V(y)\,m_{1,n}(y,k_2)|\,dy \le C\int_0^\infty \frac{|k_1-k_2|(|x|+|y|)}{1+|k_1-k_2|(|x|+|y|)}$$

$$(|x|+|y|)|V(y)|dy \le C\,\frac{|k_1-k_2||x|^2}{1+|k_1-k_2||x|} + C\int_0^\infty \frac{|k_1-k_2||y|^2}{1+|k_1-k_2||y|}dy$$

$$\le C\left[\frac{|k_1-k_2||x|}{1+|k_1-k_2||x|} + f_\gamma(k_1-k_2).\right](1+|x|). \tag{2.84}$$

Furthermore, for $x \le 0$ (see (2.7) and (2.80))

$$\int_x^0 |D_{k_1}(y-x) - D_{k_2}(y-x)|\,|V(y)\,m_{1,n}(y,k_2)|dy \le$$

$$\int_x^0 \frac{|k_1-k_2|\,(|x|+|y|)^2}{1+|k_1-k_2|(|x|+|y|)}\,|V(y)|\,(1+|y|)dy \le C\frac{|k_1-k_2||x|^2}{1+|k_1-k_2||x|}. \tag{2.85}$$

By (2.84) and (2.85) for $x \le 0$

$$|f_n(x,k_1,k_2)| \le g_\gamma(x,k_1-k_2) \tag{2.86}$$

where

$$g_\gamma(x,k) := C\left[\frac{|k||x|}{1+|k||x|} + f_\gamma(k)\right](1+|x|). \tag{2.87}$$

By (2.78) and (2.81) we have that for $x \ge 0$

$$|m_{1,n+1}(x,k_1) - m_{1,n+1}(x,k_2)| \le f_\gamma(x,k_1-k_2) + \int_x^\infty |m_{1,n}(y,k_1) - m_{1,n}(y,k_2)|\; y\,|V(y)|dy. \tag{2.88}$$

Since $m_{1,0}(x,k) \equiv 1$, it follows from (2.78) and (2.81) that

$$|m_{1,1}(x,k_1) - m_{1,1}(x,k_2)| \le f_\gamma(x,k_1-k_2),\ x \ge 0. \tag{2.89}$$

Then iterating (2.88) we prove that

$$|m_1(x,k_1) - m_1(x,k_2)| \le f_\gamma(x,k_1-k_2)\,e^{\left(\int_x^\infty y\,|V(y)|\,dy\right)},\ x \ge 0. \tag{2.90}$$

Moreover, taking the limit as $n \to \infty$ in (2.78) and using (2.21), (2.86) and (2.90) we obtain that for $x \le 0$

$$|m_1(x,k_1) - m_1(x,k_2)| \le g_\gamma(x,k_1-k_2) + \int_x^0 (|x|+|y|)|V(y)|\,|m_1(y,k_1) - m_1(y,k_2)|\,dy, \tag{2.91}$$

where in the right–hand side of (2.87) we take a constant C large enough. Let us denote

$$h(x,k_1,k_2) := \frac{|m_1(x,k_1) - m_1(x,k_2)|}{g_\gamma(x,k_1-k_2)}. \tag{2.92}$$

Then it follows from (2.91) that for $x \leq 0$

$$h(x,k_1,k_2) \leq 1 + \int_0^x (1+|y|)|V(y)| h(y,k_1,k_2) dy, \tag{2.93}$$

where we used that $g_\gamma(x,k)/(1+|x|)$ is an increasing function of $|x|$. By (2.93) and Gronwall's inequality (see page 204 of [19]) we have that

$$h(x,k_1,k_2) \leq e^{\int_0^\infty (1+|y|)|V(y)|dy} \tag{2.94}$$

and then taking in (2.87) C large enough we obtain that

$$|m_1(x,k_1) - m_1(x,k_2)| \leq g_\gamma(x,k_1-k_2). \tag{2.95}$$

We similarly prove that

$$|m_2(x,k_1) - m_2(x,k_2)| \leq g_\gamma(x,k_1-k_2). \tag{2.96}$$

Note that in the proof of (2.95), (2.96) we only used that $V \in L^1_\gamma$, $1 \leq \gamma \leq 2$. We now prove (2.71). It follows from (2.58) that

$$R_1(k_1) - R_1(k_2) = (m_1(x,k_2))^{-1} \left[e^{-2ik_1x} T(k_1)\, m_2(x,k_1) - e^{-2ik_2x} T(k_2)\, m_2(x,k_2) + \right.$$
$$\left. e^{-2ik_2x} m_1(x,-k_2) - e^{-2ik_1x} m_1(x,-k_1) + R_1(k_1)(m_1(x,k_2) - m_1(x,k_1)) \right]. \tag{2.97}$$

Then by (2.4) and (2.7) there is an $x_0 \in \mathbf{R}$ such that

$$|m_1(x,k)| \geq \frac{1}{2},\, x \geq x_0,\, k \in \mathbf{R}. \tag{2.98}$$

Then (2.71) with $j=1$ follows from (2.70), (2.95) and (2.96) taking in (2.97) any $x \geq x_0$. Equation (2.71) with $j=2$ is proven in a similar way. Equation (2.72) follows from (2.7), (2.8), (2.11), (2.12), (2.32), (2.33), (2.37), (2.38), (2.52), (2.53), the first equality in the right–hand side of (2.60) and (2.65) and noting that if $V \in L^1_2$

$$[f_1(x,k), f_2(x,k)] = ik\frac{1+a^2}{a} + O\left(k^2\right), k \to 0. \tag{2.99}$$

Equation (2.99) is proven by the argument given in [17] to prove that

$$[f_1(x,k), f_2(x,k)] = ik\frac{1+a^2}{a} + o(k), k \to 0, \tag{2.100}$$

in the case when $V \in L^1_1$. The fact that in (2.99) we have $O\left(k^2\right)$ instead of $o(k)$ follows because we assume that $V \in L^1_\gamma, \gamma \geq 2$ (see (2.11) and (2.12)). Equation (2.73) follows

from the first equality in the right–hand side of (2.60) and by (2.99). Also (2.74) follows from the first equality in the right–hand side of (2.61) and (2.62)and observing that

$$[f_1(x,k), f_2(x,-k)] = -ik\frac{a^2-1}{a} + O\left(k^2\right),\ k \to 0. \tag{2.101}$$

Equation (2.101) is proven as (2.99). It follows from (2.72) that

$$T(k_1) - T(k_2) = O\left(|k_1-k_2|^{\gamma-2}\right), k_1 - k_2 \to 0. \tag{2.102}$$

Then (2.75) with $j=1$ follows from (2.95), (2.96), (2.97) and (2.102). Equation (2.75) with $j=2$ is proven in the same way.

■

The results on the spectral theorem for H that we state below follow from the Weyl–Kodaira–Titchmarsch theory. See for example [3]. For a version of the Weyl–Kodaira–Titchmarsch theory adapted to our situation see Appendix 1 of [33] and also the proof of Theorem 6.1 in page 78 of [33]. Let us denote for any $k \in \mathbf{R}$

$$\Psi_+(x,k) := \begin{cases} \frac{1}{\sqrt{2\pi}} T(k) f_1(x,k), & k \geq 0, \\ \\ \frac{1}{\sqrt{2\pi}} T(-k) f_2(x,-k), & k < 0, \end{cases} \tag{2.103}$$

and $\Psi_-(x,-k) := \overline{\Psi_+(x,k)}$. Let $\mathcal{H}_{ac}(H)$ be the subspace of absolute continuity of H. Then the following limits

$$\hat{\phi}_\pm(k) := s - \lim_{N\to\infty} \int_{-N}^{N} \overline{\Psi_\pm(x,k)}\, \phi(x)\, dx \tag{2.104}$$

exist in the strong topology in L^2 for every $\phi \in L^2$ and the operators

$$\left(F_\pm \phi\right)(k) := \hat{\phi}_\pm(k) \tag{2.105}$$

are unitary operators from $\mathcal{H}_{ac}(H)$ onto L^2. Moreover, the ${F^*}_\pm$ are given by

$$\left({F^*}_\pm \phi\right)(x) = s - \lim_{N\to\infty} \int_{-N}^{N} \Psi_\pm(x,k)\, \phi(k)\, dk, \tag{2.106}$$

where the limits exist in the strong topology in L^2. Furthermore, the operators ${F^*}_\pm F_\pm$ are the orthogonal projection onto $\mathcal{H}_{ac}(H)$. For each eigenvalue of H, let $\Psi_j, j = 1, 2, \cdots, N$ be the corresponding eigenfunction normalized to one, i.e. $\|\Psi_j\|_{L^2} = 1$. The operators:

$$F_j \phi := (\phi, \Psi_j)\Psi_j, j = 1, 2, \cdots, N, \tag{2.107}$$

are unitary from the eigenspace generated by Ψ_j onto C. The following operators

$$F^\pm = F_\pm \oplus_{j=1}^N F_j, \tag{2.108}$$

are unitary from L^2 onto $L^2 \oplus_{j=1}^N C$ and for any $\phi \in D(H)$

$$F^\pm H\phi = \left\{k^2 (F_\pm \phi)(k), -\beta_1^2 F_1 \phi, \cdots, -\beta_N^2 F_N \phi\right\}. \tag{2.109}$$

Moreover, for any bounded Borel function , Φ, defined on $\mathbf{R}$

$$F^{\pm}\Phi(H)\phi = \left\{\Phi(k^2)(F_{\pm}\phi)(k), \Phi(-\beta_1^2)F_1\phi, \cdots, \Phi(-\beta_N^2)F_N\phi\right\}. \tag{2.110}$$

The projector, P_p, onto the subspace of L^2 generated by the eigenvectors of H is given by

$$P_p\phi := \sum_{j=1}^{N}(\phi, \Psi_j)\Psi_j. \tag{2.111}$$

Since H has no singular–continuous spectrum the projector onto the continuous subspace of H is given by: $P_c := I - P_p$. It follows from (2.110) that

$$e^{-itH}P_c = F^{\pm *}e^{-ik^2t}F^{\pm}. \tag{2.112}$$

Equation (2.112) is the starting point of our proof of the $L^1 - L^\infty$ estimate (Theorem 1.1). We divide the proof of the $L^1 - L^\infty$ estimate into a high–energy estimate and a low–energy estimate. For this purpose, let Φ be any continuous and bounded function on $\mathbf{R}$ that has a bounded derivative and such that $\Phi(k) = 0$ for $|k| \leq k_1$ and $\Phi(k) = 1$ for $|k| \geq k_2$ for some $0 < k_1 < k_2$.

LEMMA 2.6. *(The high–energy estimate). Suppose that $V \in L^1_1$. Then $e^{-itH}\Phi(H)P_c$ extends to a bounded operator from L^1 to L^∞ and there is a constant C such that*

$$\left\|e^{-itH}\Phi(H)P_c\right\|_{\mathcal{B}(L^1,L^\infty)} \leq \frac{C}{\sqrt{t}},\ t > 0. \tag{2.113}$$

Proof: Let us take $\chi \in C^\infty$, $\chi(k) = 1, |k| \leq 1$ and $\chi(k) = 0,\ k \geq 2$, and let us denote $\chi_n(k) = \chi(k/n), n = 1, 2, \cdots$. Then it follows from (2.112) that for any $f, g \in L^1 \cap L^2$:

$$\left(e^{-itH}\Phi(H)P_cf, g\right) = \lim_{n\to\infty}\left(e^{-itH}\Phi(H)\chi_n(H)P_cf, g\right) = \lim_{n\to\infty}\int dx\,dy\Phi_{t,n}(x,y)f(x)\overline{g(y)}, \tag{2.114}$$

where

$$\Phi_{t,n}(x,y) := \int_{-\infty}^{\infty} e^{-ik^2t}\chi_n(k^2)\Phi(k^2)\overline{\Psi_+(x,k)}\Psi_+(y,k)dk. \tag{2.115}$$

We have that,

$$\Phi_{t,n}(x,y) = \Phi_{t,n}^{(0)}(x,y) + \Phi_{t,n}^{(1)}(x,y) + \Phi_{t,n}^{(+)}(x,y) + \Phi_{t,n}^{(-)}(x,y), \tag{2.116}$$

where

$$\Phi_{t,n}^{(0)}(x,y) := \int_{-\infty}^{\infty} e^{-ik^2t}\,\frac{e^{-ik(x-y)}}{2\pi}\chi_n(k^2)dk, \tag{2.117}$$

$$\Phi_{t,n}^{(1)}(x,y) := \int_{-\infty}^{\infty} e^{-ik^2t}\,\frac{e^{-ik(x-y)}}{2\pi}\chi_n(k^2)(\Phi(k^2)-1)dk, \tag{2.118}$$

$$\Phi_{t,n}^{(+)}(x,y) := \int_{0}^{\infty} e^{-ik^2t}\,\frac{e^{-i(x-y)}}{2\pi}\chi_n(k^2)m_+(x,y,k)dk, \tag{2.119}$$

$$\Phi_{t,n}^{(-)}(x,y) := \int_{-\infty}^{0} e^{-ik^2t}\,\frac{e^{-ik(x-y)}}{2\pi}\chi_n(k^2)m_-(x,y,k)dk, \tag{2.120}$$

with

$$m_\pm(x,y,k) := \Phi(k^2)\left[\overline{(T(k)m_{j(\pm)}(x,k)-1)}T(k)m_{j(\pm)}(y,k)+ \right.$$
$$\left. T(k)m_{j(\pm)}(y,k)-1\right],\ \pm k \geq 0, \tag{2.121}$$

where $j(+)=1$ and $j(-)=2$. Since the inverse Fourier transform of $\frac{1}{\sqrt{2\pi}}e^{-ik^2t}$ is

$$\Phi_t^{(0)}(x) := \frac{1}{\sqrt{4\pi it}}\, e^{ix^2/4t} \tag{2.122}$$

it follows that

$$\lim_{n\to\infty}\int dx\,dy\,\Phi_{t,n}^{(0)}(x,y)f(x)\overline{g(y)} = \int dx\,dy\,\Phi_t^{(0)}(x,y)f(x)\overline{g(y)}. \tag{2.123}$$

Changing the coordinates of integration in (2.118) to $p=k-k_0$ where $k_0=(y-x)/2t$ we obtain that

$$\Phi_{t,n}^{(1)}(x,y) = \frac{1}{2\pi}e^{i(x-y)^2/4t}\int_{-\infty}^{\infty} dp e^{-ip^2t}\chi_n\left((p+k_0)^2\right)\left(\Phi\left((p+k_0)^2\right)-1\right) =$$
$$\frac{1}{2\pi\sqrt{2it}}\, e^{i(x-y)^2/4t}\int_{-\infty}^{\infty} dp\, e^{i\rho^2/4t}\,\hat{h}_n(\rho), \tag{2.124}$$

where in the second equality we used the Plancherel theorem and $\hat{h}_n(\rho)$ is the Fourier transform of the function $h_n(\rho)$ defined as follows

$$h_n(\rho) := \overline{\chi_n\left((p+k_0)^2\right)\left(\Phi\left((p+k_0)^2\right)-1\right)}. \tag{2.125}$$

Since,

$$\left\|\hat{h}_n\right\|_{L^1} \leq C\left\|h_n\right\|_{W_1} \leq C\left\|\Phi(p^2)-1\right\|_{W_1}, \tag{2.126}$$

we have that

$$\left|\Phi_{t,n}^{(1)}(x,y)\right| \leq \frac{C}{\sqrt{t}}. \tag{2.127}$$

Let us denote $h(p) := \overline{\Phi\left((p+k_0)^2\right)-1}$. Then since $\hat{h}_n(p)$ converges to $\hat{h}(p)$ in the L^1 norm, it follows from (2.124) and the dominated convergence theorem that

$$\lim_{n\to\infty}\Phi_{t,n}^{(1)}(x,y) = \Phi_t^{(1)}(x,y) := \frac{1}{2\pi\sqrt{2it}}\, e^{i(x-y)^2/4t}\int_{-\infty}^{\infty} e^{i\rho^2/4t}\,\hat{h}(\rho)d\rho, \tag{2.128}$$

and that

$$\left|\Phi_t^{(1)}(x,y)\right| \leq \frac{C}{\sqrt{t}},\ x,y\in\mathbf{R}. \tag{2.129}$$

Using the dominated convergence theorem again we prove that

$$\lim_{n\to\infty}\int dx\,dy\,\Phi_{t,n}^{(1)}(x,y)f(x)\overline{g(y)} = \int dx\,dy\,\Phi_t^{(1)}(x,y)f(x)\overline{g(y)}. \tag{2.130}$$

We denote

$$m_{+,e}(x,y,k) := \begin{cases} m_+(x,y,k), & k\geq 0,\\ \\ 0, & k<0.\end{cases} \tag{2.131}$$

Then since $\Phi(k^2)=0$ for $|k|\le\sqrt{k_1}$ and $\Phi(k^2)=1$ for $|k|\ge\sqrt{k_2}$, it follows from (2.7), (2.11), (2.63), (2.70) and (2.121) that for some constant C

$$\|m_{+,e}(x,y,\cdot)\|_{W_1}\le C,\ x,y\ge 0. \tag{2.132}$$

Then, as in the case of $\Phi_{t,n}^{(1)}$ we prove that

$$\left|\Phi_{t,n}^{(+)}(x,y)\right|\le\frac{C}{\sqrt{t}},\ x,y\ge 0,t>0. \tag{2.133}$$

and that

$$\lim_{n\to\infty}\Phi_{t,n}^{(+)}(x,y)=\Phi_t^{(+)}(x,y):=\frac{1}{2\pi\sqrt{2it}}\int_{-\infty}^{\infty}e^{i\rho^2/4t}\,\tilde m_{+,e}(x,y,\rho)d\rho, \tag{2.134}$$

where $\tilde m_{+,e}(x,y,\rho)$ is the Fourier transform of $m_{+,e}(x,y,k+k_0)$, and that

$$\left|\Phi_t^{(+)}(x,y)\right|\le\frac{C}{\sqrt{t}},\ x,y\ge 0,t>0. \tag{2.135}$$

Using (2.58) we write (2.120) as follows

$$\Phi_{t,n}^{(-)}(x,y)=\sum_{j=2}^{5}\Phi_{t,n}^{(j)}(x,y), \tag{2.136}$$

where

$$\Phi_{t,n}^{(j)}(x,y):=\int_{-\infty}^{0}e^{-ik^2t}\frac{e^{-ik(lx-ry)}}{2\pi}\chi_n(k^2)\,m_j(x,y,k)dk, \tag{2.137}$$

where for $j=2$, $l=r=3$, for $j=3$, $l=3,r=1$, for $j=4$, $l=1,r=3$, and for $j=5$, $l=r=1$. Moreover, (recall that $m_j(x,-k)=\overline{m_j(x,k)}$)

$$m_2(x,y,k):=\Phi(k^2)\left[|R_1(k)|^2\overline{m_1(x,k)}m_1(y,k)\right], \tag{2.138}$$

$$m_3(x,y,k):=\Phi(k^2)\overline{R_1(k^2)m_1(x,k)m_1(y,k)}, \tag{2.139}$$

$$m_4(x,y,k):=\Phi(k^2)R_1(k)(\overline{m_1(x,k)}-1)m_1(y,k), \tag{2.140}$$

and

$$m_5(x,y,k):=\Phi(k^2)\overline{(m_1(x,k)-1)m_1(y,k)}. \tag{2.141}$$

Then as in the case of $\Phi_{t,n}^{(+)}$ we prove that

$$\left|\Phi_{t,n}^{(-)}(x,y)\right|\le\frac{C}{\sqrt{t}},\ x,y\ge 0,t>0, \tag{2.142}$$

and that

$$\lim_{n\to\infty}\Phi_{t,n}^{(-)}(x,y)=\Phi_t^{(-)}(x,y),\ x,y\ge 0,t>0, \tag{2.143}$$

where

$$\Phi_t^{(-)}(x,y)=\sum_{j=2}^{5}\Phi_t^{(j)}(x,y), \tag{2.144}$$

with

$$\Phi_t^{(j)}(x,y) := \frac{1}{2\pi\sqrt{2t}} \int_{-\infty}^{\infty} e^{i\rho^2/4t} \tilde{m}_j(x,y,\rho)d\rho, \tag{2.145}$$

with $\tilde{m}_j(x,y,\rho)$ the Fourier transform of $m_j(x,y,p+(ry-lx)/2t)$. We also have that

$$\left|\Phi_t^{(-)}(x,y)\right| \leq \frac{C}{\sqrt{t}},\ x,y \geq 0, t > 0. \tag{2.146}$$

By the same argument as above and using also (2.59) we prove that for $(x \geq 0, y \leq 0), (x \leq 0, y \geq 0)$ and $(x \leq 0, y \leq 0)$

$$\left|\Phi_{t,n}^{(\pm)}(x,y)\right| \leq \frac{C}{\sqrt{t}},\ t > 0, \tag{2.147}$$

and that

$$\lim_{n\to\infty} \Phi_{t,n}^{(\pm)}(x,y) = \Phi_t^{(\pm)}(x,y), \tag{2.148}$$

for functions $\Phi_t^{(\pm)}(x,y)$ that satisfy

$$\left|\Phi_t^{(\pm)}(x,y)\right| \leq \frac{C}{\sqrt{t}},\ t > 0. \tag{2.149}$$

We can explicitly compute $\Phi_t^{\pm}(x,y)$ as in the case $(x \geq 0, y \geq 0)$.Then (2.147), (2.148) and (2.149) hold for all $x,y \in \mathbf{R}$ and using (2.114), (2.116), (2.123), (2.127), (2.130), (2.147) and (2.148) we prove that

$$\left(e^{-itH}\Phi(H)P_c f, g\right) = \int dx\, dy \left[\Phi_t^{(0)}(x,y) + \Phi_t^{(1)}(x,y) + \Phi_t^{(+)}(x,y) + \Phi_t^{(-)}(x,y)\right] f(x)\overline{g(y)}. \tag{2.150}$$

Then by (2.122), (2.129) and (2.149)

$$\left|\left(e^{-itH}\Phi(H)P_c f, g\right)\right| \leq \frac{C}{\sqrt{t}} \|f\|_{L^1}\|g||_{L^1},\ t > 0, \tag{2.151}$$

for all $f,g \in L^1 \cap L^2$. By continuity this estimate holds for all $f,g \in L^1$ and (2.113) follows.

■

Let Ψ be any function on $C_0^\infty(\mathbf{R})$ such that $\Psi(k) = 1, |k| \leq \delta$, for some $\delta > 0$.

LEMMA 2.7. *(The low–energy estimate). Suppose that $V \in L^1_\gamma$ where in the* generic case *$\gamma > 3/2$ and in the* exceptional case *$\gamma > 5/2$. Then $e^{-itH}\Psi(H)P_c$ extends to a bounded operator from L^1 to L^∞ and there is a constant C such that*

$$\left\|e^{-itH}\Psi(H)P_c\right\| \leq \frac{C}{\sqrt{t}},\ t > 0. \tag{2.152}$$

Proof : As in the proof of Lemma 2.6 it follows from (2.112) that for all $f, g \in L^1 \cap L^2$

$$\left(e^{-itH}\Psi(H)P_c f, g\right) = \int dx\, dy\, \Phi_t(x,y) f(x)\overline{g(y)}, \tag{2.153}$$

where

$$\Phi_t(x,y) = \Phi_t^{(+)}(x,y) + \Phi_t^{(-)}(x,y), \tag{2.154}$$

with

$$\Phi_t^{(+)}(x,y) := \int_0^\infty e^{-ik^2 t}\, \frac{e^{-ik(x-y)}}{2\pi}\, m_+(x,y,k)\, dk, \tag{2.155}$$

$$\Phi_t^{(-)}(x,y); = \int_{-\infty}^0 e^{-ik^2 t}\, \frac{e^{-ik(x-y)}}{2\pi}\, m_-(x,y,k)\, dk, \tag{2.156}$$

and

$$m_\pm(x,y,k) := \Psi(k^2) q_\pm(x,y,k) \tag{2.157}$$

with

$$q_\pm(x,y,k) := \overline{T(k) m_{j(\pm)}(x,k)} T(k) m_{j(\pm)}(y,k),\ \pm k > 0, \tag{2.158}$$

where $j(+) = 1$ and $j(-) = 2$.

Let us consider first the *generic case*. In this case it follows from (2.66) that $m_\pm(x,y,0\pm) = 0$. We denote

$$m_{+,e}(x,y,k) := \begin{cases} m_+(x,y,k), & k \geq 0, \\ \\ 0, & k < 0. \end{cases} \tag{2.159}$$

Let us denote by $\omega_{+,x,y}(\rho)$ the modulus of continuity of $m_{+,e}(x,y)$, i.e.,

$$\omega_{+,x,y}(\rho) := \|m_{+,e}(x,y,k+\rho) - m_{+,e}(x,y,k)\|_{L^2}. \tag{2.160}$$

Remark that

$$\omega_{+,x,y}(\rho) \leq 2\, \|m_{+,e}(x,y,\cdot)\|_{L^2} \leq C_{x_0},\ x,y \geq x_0. \tag{2.161}$$

Without lossing generality we can assume that $\gamma \leq 2$. Then by (2.7), (2.11), (2.70), (2.156) and (2.157) for $|\rho| \leq 1$

$$\omega_{+,x,y}(\rho) \leq C_{x_0}\, |\rho|^{\gamma-1}, x, y \geq x_0. \tag{2.162}$$

It follows from (2.161) and (2.162) that for any $0 \leq \alpha < \gamma - 1$

$$\int d\rho |\omega_{+,x,y}(\rho)|^2 \frac{1}{|\rho|^{1+2\alpha}} < \infty \tag{2.163}$$

and then by Proposition 4 in page 139 of [26]

$$\|m_{+,e}(x,y,\cdot)\|_{W_\alpha} \leq C_{\alpha,x_0}, x, y \geq x_0, \tag{2.164}$$

for any $0 < \alpha < \gamma - 1$. Let us denote $k_0 = (y-x)/2t$. Then we prove as in Lemma 2.6 that (2.127)

$$\Phi_t^{(+)}(x,y) = \frac{1}{2\pi\sqrt{2it}} \int_{-\infty}^\infty e^{i\rho^2/4t}\, \tilde{m}_{+,e}(x,y,\rho)\, d\rho, \tag{2.165}$$

with $\tilde{m}_{+,e}(x,y,\rho)$ the Fourier transform of $m_{+,e}(x,y,k+k_0)$. But since for $\frac{1}{2}<\alpha<\gamma-1$

$$\|\tilde{m}_{+,e}(x,y,\cdot)\|_{L^1} \leq C\left\|(1+\rho^2)^{\frac{\alpha}{2}}\tilde{m}_{+,e}(x,y,\cdot)\right\|_{L^2} = C\|m_{+,e}(x,y,\cdot)\|_{W_\alpha} \leq C,\ x,y\geq 0, \tag{2.166}$$

we have that

$$\left|\Phi_t^{(+)}(x,y)\right| \leq \frac{C}{\sqrt{t}},\ x,y\geq 0, t>0. \tag{2.167}$$

Using (2.7), (2.8), (2.11), (2.12), (2.58), (2.59), (2.61) and (2.71) we prove in the same way that (2.167) holds for $(x\geq 0, y<0), (x\leq 0, y\geq 0)$ and $(x\leq 0, y\leq 0)$ and that the same is true for $\Phi_t^{(-)}(x,y)$ (see the proof of Lemma 2.6 for a similar argument). Then we have that

$$|\Phi_t(x,y)| \leq \frac{C}{\sqrt{t}},\ x,y\in\mathbf{R},\ t>0. \tag{2.168}$$

Equation (2.152) follows from (2.168) as in the proof of Lemma 2.6

Let us now consider the *exceptional case*. The new problem is that now $m_\pm(x,y,0\pm)\neq 0$. Let us write $\Phi_t^{(+)}$ as follows

$$\Phi_t^{(+)}(x,y) = \Phi_t^{(1)}(x,y) + \Phi_t^{(2)}(x,y), \tag{2.169}$$

where

$$\Phi_t^{(j)}(x,y) := \int_0^\infty e^{-ik^2t}\,\frac{e^{-ik(x-y)}}{2\pi} m^{(j)}(x,y,k)\,dk,\ j=1,2, \tag{2.170}$$

with

$$m^{(1)}(x,y,k) := \Psi(k^2)\left[q_+(x,y,k)-q_+(x,y,0+)\right], \tag{2.171}$$

$$m^{(2)}(x,y,k) := \Psi(k^2)q_+(x,y,0+). \tag{2.172}$$

Then using Theorem 2.5 (b) we prove as in the *generic case* that

$$\left|\Phi_t^{(1)}(x,y)\right| \leq \frac{C}{\sqrt{t}},\ x,y\in\mathbf{R},\ t>0. \tag{2.173}$$

Let $\hat{\Psi}(\lambda), \lambda\geq 0$, be the cosine transform of $\Psi(k^2)$:

$$\hat{\Psi}(\lambda) := \int_0^\infty \cos(\lambda k)\Psi(k^2)dk. \tag{2.174}$$

Then integrating by parts we prove that for any $N>0$ there is a constant C_N such that

$$\left|\hat{\Psi}(\lambda)\right| \leq C_N\,(1+|\lambda|)^{-N}. \tag{2.175}$$

Since

$$\Psi(k^2) = \frac{2}{\pi}\int_0^\infty \cos(\lambda k)\hat{\Psi}(\lambda)d\lambda, \tag{2.176}$$

we have that

$$\Phi_t^{(2)}(x,y) = \frac{q_+(x,y,0+)}{\pi}\int_0^\infty d\lambda\hat{\Psi}(\lambda)\int_0^\infty e^{-ik^2t}e^{-ik(x-y)}\cos(\lambda k)\,dk. \tag{2.177}$$

But

$$\left|\int_0^\infty e^{-ik^2 t} e^{-ik(x-y)} \cos(\lambda k)\, dk\right| \leq \frac{C}{\sqrt{t}},\ t > 0. \tag{2.178}$$

The estimate (2.178) is proven by explicitly evaluating the cosine transform using the following equations from [4]: 3 in page 7, 1 in page 23, 7 in page 24, 3 in page 63, 1 in page 82 and 3 in page 83. Then by (2.175), (2.177) and (2.178)

$$\left|\Phi_t^{(2)}(x,y)\right| \leq \frac{C}{\sqrt{t}},\ x, y \in \mathbf{R},\ t > 0. \tag{2.179}$$

It follows from (2.169), (2.173) and (2.179) that

$$\left|\Phi_t^{(+)}(x,y)\right| \leq \frac{C}{\sqrt{t}},\ x, y \in \mathbf{R},\ t > 0. \tag{2.180}$$

We prove in the same way that

$$\left|\Phi_t^{(-)}(x,y)\right| \leq \frac{C}{\sqrt{t}},\ x, y \in \mathbf{R},\ t > 0. \tag{2.181}$$

Equation (2.152) follows from (2.153), (2.154), (2.180) and (2.181) as in the *generic case*.
Proof of Theorem 1.1: The theorem follows from Corollaries 2.6 and 2.7.
Proof of Corollary 1.2: Since H is self–adjoint

$$\left\|e^{-itH} P_c\right\|_{\mathcal{B}(L^2)} \leq 1. \tag{2.182}$$

Then the corollary follows interpolating between (1.17) and (2.182) (see the Appendix to [24]).
Proof of Corollary 1.3: Corollary 1.3 follows from Corollary 1.2 as in the proof of Theorem 4.1 of [14].

3 Inverse Scattering

Proof of Theorem 1.4: We prove this theorem by verifying the conditions of the abstract Theorems 1 and 2 of [27] and of Theorem 16 of [28]. This is done as in Theorem 8 of [27] and Theorem 17 of [28]. We define X and X_3 as in the Introduction and $X_1 := L^{1+\frac{1}{p}}$. It follows from the Sobolev imbedding theorem (see [1]) that $X \subset X_3$, with continuous imbedding. Concerning hyphotesis (V) in page 113 of [27]: note that since by Sobolev's imbedding theorem $W_1 \subset L^{1+p}$; we have that $X_1 \subset W_1$. But as $e^{-itH} \in \mathcal{B}(W_1)$, it follows by duality that $e^{-itH} \in \mathcal{B}(W_{-1})$. Then for all $\phi \in X_1$, $e^{-itH}\, \phi \in W_{-1}$ and $e^{-itH}\, e^{-isH}\phi = e^{-i(t+s)H}\phi$ for all $t, s \in \mathbf{R}$.

To verify hypothesis VII of Theorem 16 of [28], as in the proof of Theorem 8 of [27], we need the following result. Let g be any real–valued C^2 function defined on $\mathbf{R}$ such that $g(0) = 0$ and for all $u, v \in \mathbf{R}$:

$$|g(u) - g(v)| + |\acute{g}(u) - \acute{g}(v)| \leq C|u - v|, \tag{3.1}$$

and

$$|g(u)| \leq |f(u)|. \tag{3.2}$$

For I any interval let us denote by $C(I,X)$ the Banach space of bounded and continuous functions from I into X with the supremun norm and by $B_\rho(I,X)$ the ball of center zero and radius ρ in $C(I,X)$. Then for any $\phi \in X(\rho/2)$ and any $s \in \mathbf{R}$ the equation

$$u(t) = e^{-itH}\phi + \frac{1}{i}\int_s^t e^{-i(t-\tau)H} P_g(u(\tau))d\tau, \tag{3.3}$$

where

$$P_g(u(\tau)) := g(|u(\tau)|)\frac{u(\tau)}{|u(\tau)|} \tag{3.4}$$

has a unique solution $u(t) \in B_\rho(\mathbf{R}, X)$ and moreover, the L^2 norm and the energy are conserved:

$$\|u(t)\|_{L^2} = \text{constant} \tag{3.5}$$

$$E_g := \frac{1}{2}\|\sqrt{H}u(t)\|_{L^2}^2 + \int dx G(|u(t)|) = \text{constant}, \tag{3.6}$$

for all $t \in \mathbf{R}$, where G is the primitive of g such that $G(0) = 0$. To prove this result we observe that it follows from (3.1) and (3.2) that

$$\|P_g(\phi) - P_g(\psi)\|_X \leq C\left(\|\phi\|_X + \|\psi\|_X\right)\|\phi - \psi\|_X, \tag{3.7}$$

for all $\phi, \psi \in X$. Then by a standard contraction mapping argument (3.3) as a unique solution on $C([s-\epsilon, s+\epsilon], X)$ provided that $0 < \epsilon \leq 1/3C\rho$ and $0 < \epsilon < 1/2C$. Suppose that (3.5) and (3.6) are true for $t \in [s-\epsilon, s+\epsilon]$. Then since $|G(\lambda)| \leq C\lambda^2$,

$$\|u(t)\|_X^2 \leq 2E_g + 2(1+C)\|u(t)\|_{L^2}^2 \leq C,\ t \in [s-\epsilon, s+\epsilon]. \tag{3.8}$$

Since $\|u(t)\|_X$ remains bounded as $t \to s \pm \epsilon$ by a constant C that depends only on $\|\phi\|_X$ we can extend $u(t)$ into a global solution such that (3.5), (3.6) hold for all $t \in \mathbf{R}$. It remains to prove that (3.5), (3.6) are true for $t \in [s-\epsilon, s+\epsilon]$. In the constant coefficient case, $V = 0$, this is accomplished by approximating the local solution in W_1 by solutions in W_2, see [15] and [16] or by regularizing equation (3.3) by taking convolution with a function in Schwartz space, see [7], [8] and [9]. This is possible because in the constant coefficient case $D(H) = D(\Delta) = W_2$. In our case this is not a convenient approach. Since we only assume that $V \in L^1_\gamma$ we do not have much control over $D(H)$. We only know that $D(H)$ is a dense set in X. To solve this problem we regularize (3.3) multiplying it by an appropriate function of H. Let us denote $r_n(H) := \left(\frac{H}{n} + 1\right)^{-1}, n = 1, 2, \cdots$. The regularized equation is given by

$$u_n(t) = e^{-itH} r_n(H)\phi + \frac{1}{i}\int_s^t e^{-i(t-\tau)H} r_n(H) P_g\left(r_n(H)u_n(\tau)\right)\, d\tau. \tag{3.9}$$

As above we prove that (3.9) has a unique solution for $t \in [s-\epsilon, s+\epsilon]$. Note that we can take ϵ independent on n. Moreover, since $Hr_n(H) \in \mathcal{B}(X)$ we have that actually $u_n(t) \in C^1([s-\epsilon, s+\epsilon], X)$. Then

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = 2\Re\left(u_n(t), \frac{\partial}{\partial t}u_n(t)\right). \tag{3.10}$$

Since $u_n(t)$ is a solution to the equation

$$i\frac{\partial}{\partial t}u_n(t) = Hu_n(t) + r_n(H)g\left(|r_n(H)u_n(t)|\right)\frac{r_n(H)u_n(t)}{|r_n(H)u_n(t)|} \tag{3.11}$$

and since H is self–adjoint, it follows from (3.10) that

$$\frac{d}{dt}\|u_n(t)\|_{L^2}^2 = 0. \tag{3.12}$$

Furthermore,

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{H}u_n(t)\|_{L^2}^2 = \Re\left(\sqrt{H}u_n(t), \sqrt{H}\frac{\partial}{\partial t}u_n(t)\right). \tag{3.13}$$

Let us define

$$Q_n(t) := \int dx\, G\left(|r_n(H)u_n|\right). \tag{3.14}$$

Since $|G(\lambda)| \leq C|\lambda|^2$,

$$|Q_n(t)| \leq C\|u_n(t)\|_{L^2}^2. \tag{3.15}$$

Furthermore, since $u_n(t) \in C^1([s-\epsilon, s+\epsilon], X)$ it follows from a simple proof using the fundamental theorem of calculus (see the proof of Lemma 3.1 of [7] for a similar argument) that

$$\frac{d}{dt}Q_n(t) = \Re\left(r_n(H)\frac{g(|r_n(H)u_n(t)|)}{|r_n(H)u_n(t)|}r_n(H)u_n(t), \frac{\partial}{\partial t}u_n(t)\right). \tag{3.16}$$

We define the regularized energy as follows

$$E_n(t) := \frac{1}{2}\|\sqrt{H}u_n(t)\|_{L^2}^2 + Q_n(t). \tag{3.17}$$

It follows from (3.11), (3.13), (3.16) and since H is self–adjoint that

$$\frac{d}{dt}E_n(t) = 0. \tag{3.18}$$

By (3.12) and (3.18), $\|u_n(t)\|_{L^2}$ and $E_n(t)$ are constant for $t \in [s-\epsilon, s+\epsilon]$. We prove below that $u_n(t)$ converges strongly in X to $u(t)$. Since moreover, $r_n(H)$ converges to the identity strongly in X, equations (3.5) and (3.6) hold for $t \in [s-\epsilon, s+\epsilon]$. It only remains to prove that

$$\lim_{n\to\infty}\|u_n(t) - u(t)\|_X = 0. \tag{3.19}$$

But by (3.3), (3.7) and (3.9)

$$\|u_n(t) - u(t)\|_X \leq \int_s^t d\tau\, \|r_n(H)P_g(r_n(H)u_n) - r_n(H)P_g(r_n(H)u)\|_X +$$

$$\int_s^t d\tau \|r_n(H)P_g(r_n(H)u) - P_g(u)\|_X \leq 2C\epsilon\,\rho\|u_n - u\|_{C([s-\epsilon,s+\epsilon],X)} +$$

$$2C\epsilon\,\rho \int_s^t \|(r_n(H) - 1)u(\tau)\|_X\, d\tau. \tag{3.20}$$

But since $2C\epsilon\rho < 2/3$

$$\|u_n - u\|_{C([s-\epsilon,s+\epsilon],X)} \leq 6C\epsilon\rho \int_{s-\epsilon}^{s+\epsilon} \|(r_n(H)-1)u(\tau)\| d\tau \to 0, \tag{3.21}$$

as $n \to \infty$. As in the proof of Theorem 17 of [28] we have to prove that $e^{-itH} \in \mathcal{B}\left(X, L^r\left(\mathbf{R}, L^{1+p}\right)\right)$. Let us denote by $\mathcal{D}$ the set of points in the $(\frac{1}{q}, \frac{1}{r})$ plane, $1 \leq p, q \leq \infty$, such that $e^{-itH} \in \mathcal{B}\left(X, L^r\left(\mathbf{R}, L^q\right)\right)$. We already know that $A := (\frac{1}{2}, 0) \in \mathcal{D}$ because e^{-itH} is a unitary operator on L^2. Since e^{-itH} is unitary on X, we have that $e^{-itH} \in \mathcal{B}\left(X, L^\infty(X)\right)$ and as by Sobolev's theorem [1] X is continuously embedded in L^∞ it follows that $B := (0,0) \in \mathcal{D}$. By Corollary 1.3 $e^{-itH} \in \mathcal{B}\left(L^2, L^6\left(\mathbf{R}, L^6\right)\right)$ and then $C := (\frac{1}{6}, \frac{1}{6}) \in \mathcal{D}$. Since $A, B, C \in \mathcal{D}$ it follows by interpolation (see [24]) that the solid triangle with vertices A, B, C belongs to $\mathcal{D}$. Let us consider the following curve, $\mathcal{C}$, in the $(\frac{1}{q}, \frac{1}{r})$ plane:

$$\frac{1}{r} := \left(\frac{1}{2} + \frac{1}{q}\right) / (q-2) = -\frac{1}{2} + \frac{1}{2 - \frac{4}{q}} - \frac{1}{2q},\ 1 \leq q \leq 6. \tag{3.22}$$

Note that $\mathcal{C}$ goes from B to C and that for $0 \leq \frac{1}{q} \leq \frac{1}{6}$ the curve $\mathcal{C}$ is contained in the triangle with vertices (A, B, C). Then $\mathcal{C} \subset \mathcal{D}$ for $0 \leq \frac{1}{q} \leq \frac{1}{6}$ and then taking $q = p-1$, we have that $e^{-itH} \in \mathcal{B}\left(X, L^r\left(L^{p+1}\right)\right)$ for $5 \leq p \leq \infty$, with $r := (p-1)/(1-d)$.

Proof of Theorem 1.5 : The proof of Theorem 1.1 of [31] applies in our case with no changes.

Proof of Corollary 1.6 : By Theorem 1.5 S determines uniquely S_L. Let us denote

$$\hat{S}_L := F S_L F^* \tag{3.23}$$

and let U be the following unitary operator from L^2 onto $L^2(\mathbf{R}^+) \oplus L^2(\mathbf{R}^+)$:

$$Uf(k) := \left\{ \begin{array}{c} f_1(k) \\ f_2(k) \end{array} \right\}, \tag{3.24}$$

where $f_1(k) := f(k), k \geq 0$, and $f_2(k) := f(-k), k \geq 0$. Let us denote

$$\tilde{S}_L := U \hat{S}_L U^*. \tag{3.25}$$

Pearson proved in Section 9.7 of [22] that for V bounded and with fast decay:

$$\tilde{S}_L \left\{ \begin{array}{c} f_1(k) \\ f_2(k) \end{array} \right\} = \left[\begin{array}{cc} T(k) & R_1(k) \\ R_2(k) & T(k) \end{array} \right] \left[\begin{array}{c} f_1(k) \\ f_2(k) \end{array} \right]. \tag{3.26}$$

Let us assume that $V \in L^1_\delta$ for some $\delta > 1$. Let $V_n \in C_0^\infty, n = 1, 2, \cdots$ be such that

$$\lim_{n\to\infty} \|V_n - V\|_{L^1_\delta} = 0. \tag{3.27}$$

Let us denote by $S_{L,n}, T_n(k)$ and $R_{j,n}(k), j = 1, 2$, the scattering operator, the transmission coefficient and the reflection coefficients corresponding to V_n. Then by the proof of Lemma 1 of [3] and by equations (2.60) to (2.62)

$$\lim_{n\to\infty} T_n(k) = T(k),\ \lim_{n\to\infty} R_{j,n}(k) = R_j(k), j = 1, 2. \tag{3.28}$$

Moreover, by the stationary formula for the wave operators (see equation (12.7.5) of [25]) and from the results in Chapter 12 of [25]

$$s - \lim_{n \to \infty} S_{L,n} = S_L, \tag{3.29}$$

where the limit exists in the strong topology in L^2. Then by continuity (3.26) is true also for $V \in L^1_\delta, \delta > 1$ and it follows that from S_L we obtain the transmission coefficient and the reflection coefficients. But since V has no bound states one of the reflection coefficients uniquely determines V (see for example [5], [6], [3], [20] [2] or [10]).
Proof of Corollary 1.7: The proof of Corollary 1.3 of [31] applies in this case with no changes.

References

[1] R. A. Adams, "Sobolev Spaces", Academic Press, New York, 1975.

[2] K. Chadan and P.C. Sabatier, "Inverse Problems in Quantum Scattering Theory. Second Edition", Springer–Verlag, Berlin, 1989.

[3] P. Deift and E. Trubowitz, Inverse scattering on the line, Comm. Pure Appl. Math. **XXXII** (1979), 121–251.

[4] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, "Tables of Integral Transforms, Volume I", McGraw–Hill, New York, 1974.

[5] L. D. Faddeev, Properties of the S matrix of the one–dimensional Schrödinger equation, Trudy Math. Inst. Steklov **73** (1964), 314–333 [english translation American Mathematical Society Translation Series 2 **65** (1964),139–166].

[6] L. D. Faddeev, Inverse problems of quantum scattering theory, II, Itogi Nauki i Tekhniki Sovremennye Problemy Matematiki **3** (1974), 93–180 [english translation J. Soviet Math. **5** (1976), 334–396].

[7] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case, J. Funct. Anal. **32** (1979), 1–32.

[8] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equation. II. Scattering theory, general case, J. Funct. Anal. **32** (1979), 33–71.

[9] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations. III. Special theories in dimensions 1, 2 and 3, Ann. Inst. H. Poincaré Phys. Théor. **XXVIII** (1978), 287–316.

[10] B. Grebert and R. Weder, Reconstruction of a potential on the line that is a priori known on the half–line, SIAM J. Appl. Math. **55** (1995), 242–254.

[11] A. Jensen, Spectral properties of Schrödinger operators and time–decay of the wave functions. Results in $L^2(\mathbf{R}^m), m \geq 5$, Duke. Math. J. **47** (1980), 57–80.

[12] A. Jensen, Spectral properties of Schrödinger operators and time–decay of the wave functions. Results in $L^2(\mathbf{R}^4)$, J. Math. Anal. Appl. **101** (1984), 397–422.

[13] A. Jensen and T. Kato, Spectral properties of Schrödinger operators and time–decay of the wave functions, Duke Math. J. **46** (1979), 583–611.

[14] J. L. Journé, A. Soffer and C. D. Sogge, Decay estimates for Schrödinger operators, Comm. Pure Appl. Math. XLIV (1991), 573–604.

[15] T. Kato, On nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Phys. Théor. **46** (1987), 113–129.

[16] T. Kato, Nonlinear Schrödinger equations, in "Schrödinger Operators" Lecture Notes in Physics **345**, pp. 218–263. Editors H. Holden and A. Jensen, Springer–Verlag, Berlin, 1989.

[17] M. Klaus, Low–energy behaviour of the scattering matrix for the Schrödinger equation on the line, Inverse Problems **4** (1988), 505–512.

[18] Li Ta–Tsien and Chen Yunmei, "Global Solutions for Nonlinear Evolution Equations", Longman Scientific & Technical, Harlow, 1992.

[19] R. H. Martin Jr., "Nonlinear Operators and Differential Equations in Banach Spaces", John Wiley & Sons, New York, 1976.

[20] A. Melin, Operator methods for inverse scattering on the real line, Comm. Part. Differential Equations **10** (1985), 677–766.

[21] R. G. Newton, Low–energy scattering for medium range potentials, J. Math. Phys. **27** (1986), 2720–2730.

[22] D. B. Pearson, "Quantum Scattering and Spectral Theory", Academic Press, New York, 1988.

[23] R. Racke, "Lectures in Nonlinear Evolution Equations. Initial Value Problems", Aspects of Mathematics **E 19**,F. Vieweg & Son, Braunschweig/Wiesbaden, 1992.

[24] M. Reed and B. Simon, "Methods of Modern Mathematical Physics II. Fourier Analysis, Self–Adjointness, Academic Press, New York, 1978.

[25] M. Schechter, "Operator Methods in Quantum Mechanics", North Holland, New York, 1981.

[26] E. M. Stein, "Singular Integrals and Differentiability Properties of Functions", Princeton Univ. Press, Princeton, 1970.

[27] W. A. Strauss, Nonlinear scattering theory at low energy, J. Funct. Anal. **41** (1981), 110–133.

[28] W. A. Strauss, Nonlinear scattering theory at low energy: sequel, J. Funct. Anal. **43** (1981),281–293.

[29] W. A. Strauss, “Nonlinear Wave Equations”, CBMS–RCSM **73**, American Mathematical Society, Providence, 1989.

[30] R. S. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, Duke Math. J. **44** (1977), 705–714.

[31] R. Weder, Inverse scattering for the nonlinear Schrödinger equation, Commun. Partial Differential Equations **22** (1997), 2089–2103.

[32] J. Weidmann, “Spectral Theory of Ordinary Differential Operators”, Lecture Notes in Mathematics **1258**, Springer–Verlag, Berlin, 1987.

[33] C. H. Wilcox, “Sound Propagation in Stratified Fluids”, Applied Mathematical Sciences **50**, Springer–Verlag, New York, 1984.